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Quadruple Yukawa Unification in the Minimal Supersymmetric Standard Model

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Abstract

Predictions for m_t , $\tan \beta$, m_{ν_τ} are calculated for quadruple third family $t - b - \tau - \nu_\tau$ Yukawa unified models in the MSSM. The renormalisation group equations for the 3 families of the MSSM, including the right handed neutrino, are presented. For right handed tau neutrino Majorana masses that are bigger than 10^{11} GeV, the tau neutrino mass is consistent with present cosmological bounds. The m_t , $\tan \beta$ predictions are approximately equivalent to those in triple third family Yukawa unified models.

The origin of the Yukawa couplings of the quarks and leptons is one of the major puzzles facing the standard model. In the context of supersymmetry (SUSY), and in particular the minimal supersymmetric standard model (MSSM), some simplifications may occur. The reason for this is simply that the low-energy gauge couplings, when evolved using the renormalisation group equations (RGE's) of the MSSM, converge at a scale $M_X = 10^{16} - 10^{17}$ GeV, which hints at some further stage of unification at this scale. Unified or partially unified gauge groups often constrain not only the gauge couplings but also the Yukawa couplings of the theory, and offer the possibility of understanding the low-energy Yukawa couplings in terms of some simple pattern of Yukawa couplings at the high-energy scale M_X .

It was realized some time ago that the simplest grand unified theories (GUTs) based on $SU(5)$ predict the Yukawa couplings of the bottom quark and the tau lepton to be equal at the GUT scale [1],

$$\lambda_b(M_X) = \lambda_\tau(M_X) \quad (1)$$

where $M_X \sim 10^{16}$ GeV. Assuming the effective low energy theory below M_{GUT} to be that of the minimal supersymmetric standard model (MSSM) the boundary condition in Eq.1 leads to a physical bottom to tau mass ratio m_b/m_τ that is in good agreement with experiment [2]. Spurred on by recent LEP data which is consistent with coupling constant unification, the relation in Eq.1 has recently been the subject of intense scrutiny using increasingly sophisticated levels of approximation [3, 4].

One may take this idea a stage further and consider theories based on $SO(10)$ or $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ [5, 6, 7] which predict the Yukawa couplings of the top quark, bottom quark and tau lepton all to be equal at the unification scale,

$$\lambda_t(M_X) = \lambda_b(M_X) = \lambda_\tau(M_X). \quad (2)$$

In such theories the top and bottom Yukawa couplings run almost identically down to low energies, and the observed mass splitting between the top and bottom is ascribed to a large ratio of vacuum expectation values (VEVs) of the two Higgs doublets of the MSSM, where the Higgs doublet which gains the large VEV couples to the top quark, and the Higgs doublet which gains the small VEV couples to the bottom quark. This ratio of VEVs is conventionally defined to be $\tan \beta \equiv v_2/v_1$, so that top-bottom-tau Yukawa unification predicts large $\tan \beta \approx m_t/m_b$.

In this letter we shall include the tau neutrino Yukawa coupling, which is predicted in some models to be equal to that of the other third family Yukawa couplings at the

unification scale M_X :

$$\lambda_t(M_X) = \lambda_b(M_X) = \lambda_\tau(M_X) = \lambda_{\nu_\tau}(M_X) \equiv \lambda_{33}(M_X). \quad (3)$$

We refer to this as quadruple Yukawa unification. As we shall see, the extra neutrino couplings can influence the predictions which follow from top-bottom-tau Yukawa unification if the right-handed neutrinos have a Majorana mass $M < M_X$. Such right-handed neutrino Majorana masses may occur in the two-loop Witten mechanism, for example [8].

The trilinear superpotential for the MSSM including a right handed neutrino is

$$W = Y^U u Q H_2 + Y^D d Q H_1 + Y^E e L H_1 + Y^N \nu L H_2 + \text{h.c.}, \quad (4)$$

where Y^U , Y^D , Y^E and Y^N label the up quark, down quark, charged lepton and neutrino Yukawa matrices respectively¹. When the Higgs particles obtain their vacuum expectation values (VEVs) $\langle H_1 \rangle = v \cos \beta$ and $\langle H_2 \rangle = v \sin \beta$, the neutrinos acquire Dirac masses m^D .

The one loop coupling evolution of the Yukawa couplings in Eq.4 was calculated in the \overline{MS} renormalisation scheme, using an analysis of general superpotentials performed by Martin and Vaughn [9]:

$$\begin{aligned} \frac{\partial g_i}{\partial t} &= \frac{b_i g_i^3}{16\pi^2} \\ \frac{\partial Y^U}{\partial t} &= \frac{Y^U}{16\pi^2} \left[\text{Tr} \left(3Y^U Y^{U\dagger} + Y^{N\dagger} Y^N \right) + 3Y^{U\dagger} Y^U + Y^{D\dagger} Y^D - \left(\frac{13}{15} g_1^2 + 3g_2^2 + \frac{16}{3} g_3^2 \right) \right] \\ \frac{\partial Y^D}{\partial t} &= \frac{Y^D}{16\pi^2} \left[\text{Tr} \left(3Y^D Y^{D\dagger} + Y^E Y^{E\dagger} \right) + Y^{U\dagger} Y^U + 3Y^{D\dagger} Y^D - \left(\frac{7}{15} g_1^2 + 3g_2^2 + \frac{16}{3} g_3^2 \right) \right] \\ \frac{\partial Y^E}{\partial t} &= \frac{Y^E}{16\pi^2} \left[\text{Tr} \left(3Y^E Y^{E\dagger} + Y^D Y^{D\dagger} \right) + 3Y^{E\dagger} Y^E + Y^{N\dagger} Y^N - \left(\frac{9}{5} g_1^2 + 3g_2^2 \right) \right] \\ \frac{\partial Y^N}{\partial t} &= \frac{Y^N}{16\pi^2} \left[\text{Tr} \left(3Y^N Y^{N\dagger} + Y^U Y^{U\dagger} \right) + 3Y^{N\dagger} Y^N + Y^{E\dagger} Y^E - \left(\frac{3}{5} g_1^2 + 3g_2^2 \right) \right], \end{aligned} \quad (5)$$

where $b_i = (33/5, 1, -3)$, $t = \ln \mu$ and μ is the \overline{MS} renormalisation scale.

The see saw mechanism assumes the neutrino mass terms are of the form

$$\left[\overline{(\nu_L)} \overline{(\nu_R)^c} \right] \left[\begin{array}{cc} 0 & m^D/2 \\ (m^D)^T/2 & M \end{array} \right] \left[\begin{array}{c} (\nu_L)^c \\ (\nu_R) \end{array} \right] + \text{h.c.} = m^D \overline{(\nu_L)} (\nu_R) + M \overline{(\nu_R)^c} (\nu_R) + \text{h.c.}, \quad (6)$$

where $M \gg m^D$ is the Majorana mass of the right handed neutrino. In general, m^D and M are 3 by 3 matrices in family space. In quadruple Yukawa unified models, the

¹Family and gauge indices have been suppressed in Eq.4.

Dirac masses m^D of ν_e and ν_μ are several orders of magnitude smaller than those of the third family, similar to the charged fermions. It is therefore a good approximation to consider the third family alone and drop smaller Yukawa couplings.

Once the small couplings have been dropped, Eqs.5 reduce to the RGEs derived in [10]:

$$\begin{aligned}
16\pi^2 \frac{\partial g_i}{\partial t} &= b_i g_i^3 \\
16\pi^2 \frac{\partial \lambda_t}{\partial t} &= \lambda_t \left[6\lambda_t^2 + \lambda_b^2 + \theta_R \lambda_{\nu_\tau}^2 - \left(\frac{13}{15} g_1^2 + 3g_2^2 + \frac{16}{3} g_3^2 \right) \right] \\
16\pi^2 \frac{\partial \lambda_b}{\partial t} &= \lambda_b \left[6\lambda_b^2 + \lambda_\tau^2 + \lambda_t^2 - \left(\frac{7}{15} g_1^2 + 3g_2^2 + \frac{16}{3} g_3^2 \right) \right] \\
16\pi^2 \frac{\partial \lambda_\tau}{\partial t} &= \lambda_\tau \left[4\lambda_\tau^2 + 3\lambda_b^2 + \theta_R \lambda_{\nu_\tau}^2 - \left(\frac{9}{5} g_1^2 + 3g_2^2 \right) \right] \\
16\pi^2 \frac{\partial \lambda_{\nu_\tau}}{\partial t} &= \lambda_{\nu_\tau} \left[4\theta_R \lambda_{\nu_\tau}^2 + 3\lambda_t^2 + \lambda_\tau^2 - \left(\frac{3}{5} g_1^2 + 3g_2^2 \right) \right], \tag{7}
\end{aligned}$$

where $\theta_R \equiv \theta(t - \ln M)$ takes into account the large mass suppression of the right-handed neutrino loops at scales $\mu < M$. Thus we integrate out loops involving right-handed neutrinos at M , but retain the Dirac Yukawa coupling λ_{ν_τ} which describes the coupling of left to right-handed neutrinos. The running procedure to determine the low energy masses is then to run down the (Dirac) neutrino Yukawa coupling λ_{ν_τ} from M_X to low-energies using the above RGEs. Then at low-energies we use the usual see-saw mechanism to determine the mass of the physical light tau neutrino.

The procedure to extract the predictions from quadruple Yukawa unification is as follows. Values of M , $m_b(m_b)$, $\alpha_s(M_Z)$ and M_X (the unification scale) were chosen as free parameters. Values of a scale Λ and $\lambda_{33}(M_X)$ are taken. The gauge couplings g_i are determined at M_X by using the input values $\alpha_1(M_Z)^{-1} = 58.89$, $\alpha_2(M_Z)^{-1} = 29.75$ and $\alpha_3(M_Z) = 0.10 - 0.13$. The gauge couplings were then run to Λ by using the standard model RG equations including 5 quark flavours and no scalar fields. The whole superparticle spectrum of the MSSM is assumed to lie at Λ as an approximation and the gauge couplings are determined at M_X by using the RGEs for the MSSM [4]. Although exact gauge unification is not imposed, it is nevertheless approximately realised. Λ is taken to be $m_t \sim M_{susy}$ and Eq.3 is imposed as a boundary condition². The third family and gauge couplings are evolved from M_X to Λ using Eq.s7.

The various couplings at Λ were determined as follows. The running masses of the fermions $m_{b,\tau}(\Lambda)$, were determined by running them up from their mass shell values

²Note that in all of the predictions, altering M_{susy} to 1 TeV makes only a negligible difference and so we are justified in taking $M_{susy} = m_t$.

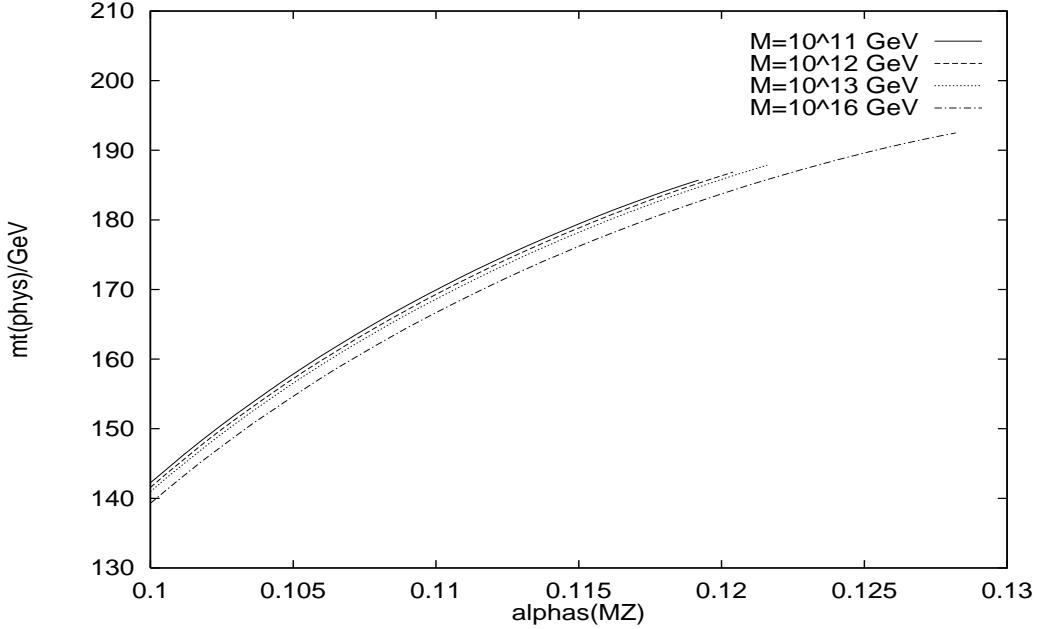


Figure 1: Physical m_t predicted for various M , $M_X = 10^{16}$ GeV and $m_b = 4.25$ GeV.

$m_f(m_f)$ with effective 3 loop QCD \otimes 1 loop QED [11, 12, 13, 14]. This enables us to calculate $\cos \beta$ at Λ from the $\lambda_\tau(\Lambda)$ predicted by quadruple Yukawa unification:

$$\cos \beta = \frac{\sqrt{2}m_\tau(\Lambda)}{v\lambda_\tau(\Lambda)}. \quad (8)$$

The determination of $\cos \beta$ allows us to make the following mass predictions from the Yukawa couplings calculated at the scale Λ ,

$$\begin{aligned} m_t(\Lambda) &= \frac{\sqrt{2}v\lambda_t(\Lambda)}{\sin \beta} \\ m_b(\Lambda) &= \frac{\sqrt{2}v\lambda_b(\Lambda)}{\cos \beta} \\ m_{\nu_\tau}(\Lambda) &= \frac{\sqrt{2}v\lambda_{\nu_\tau}(\Lambda)}{\sin \beta} \end{aligned} \quad (9)$$

Values of Λ and $\lambda_{33}(M_X)$ are searched through until $m_t = \Lambda$ is a prediction of Eq.9, and $m_b(\Lambda)$ is predicted by Eq.9 to be the empirically derived value obtained by running $m_b(\mu)$ up from $m_b(m_b)$ as explained above. The light tau neutrino mass can now be predicted by diagonalising Eq.6 in the one family case and extracting the small mass eigenvalue:

$$m_{\nu_\tau}(m_{\nu_\tau}) = \frac{m_{\nu_\tau}^D(m_t)^2}{4M}. \quad (10)$$

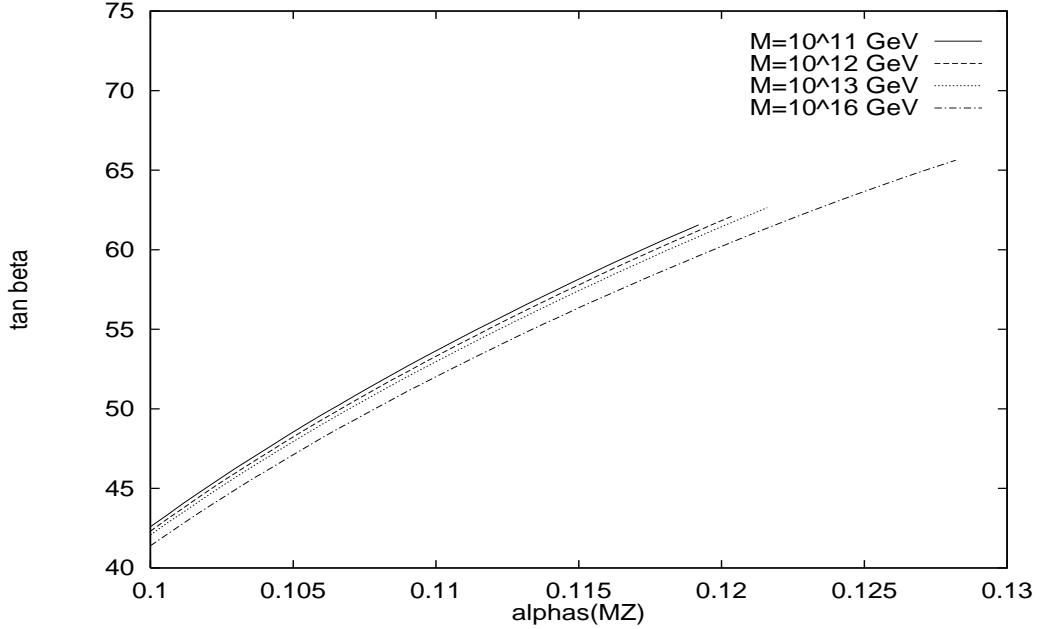


Figure 2: $\tan \beta$ predicted for various M and $M_X = 10^{16}$ GeV and $m_b = 4.25$ GeV.

This equation is approximately correct because the Dirac mass does not renormalise significantly from m_t to M_Z and below that scale, effective QCD \otimes QED does not renormalise neutrino masses. We thus have a prediction for m_t but whereas the m_t referred to here is the running one, it can be related as in [4] to the physical mass by

$$m_t^{phys} = m_t(m_t) \left[1 + \frac{4}{3\pi} \alpha_3(m_t) + O(\alpha_3^2) \right]. \quad (11)$$

Fig.1 displays the difference in the m_t predictions³ for different M . $M = 10^{16}$ GeV corresponds to integrating out the right handed neutrino at M_X and so reduces to the previously studied case of triple Yukawa unification in the MSSM [15, 16]. As the figure shows, including the right handed neutrino Yukawa coupling makes only a small difference of up to 3 GeV to m_t . m_t is insensitive to whether the unification scale is 10^{16} or 10^{17} GeV. Experimentally derived errors on $m_b, \alpha_S(M_Z)$ provide a much larger variation in m_t . Fig.2 shows the difference in the $\tan \beta$ prediction when the right handed neutrino is included. Again, only a small deviation from the triple Yukawa unification prediction is observed. m_b and $\alpha_S(M_Z)$ uncertainties provide a larger deviation in the prediction [6]. $\tan \beta$ is also insensitive to whether $M_X = 10^{16}$

³Note that in Fig.s 1-4 the curves stop short for differing $\alpha_S(M_Z)$ because at these values, quadruple Yukawa unification is not possible with perturbative Yukawa couplings (these are constrained to be less than 5.0.)

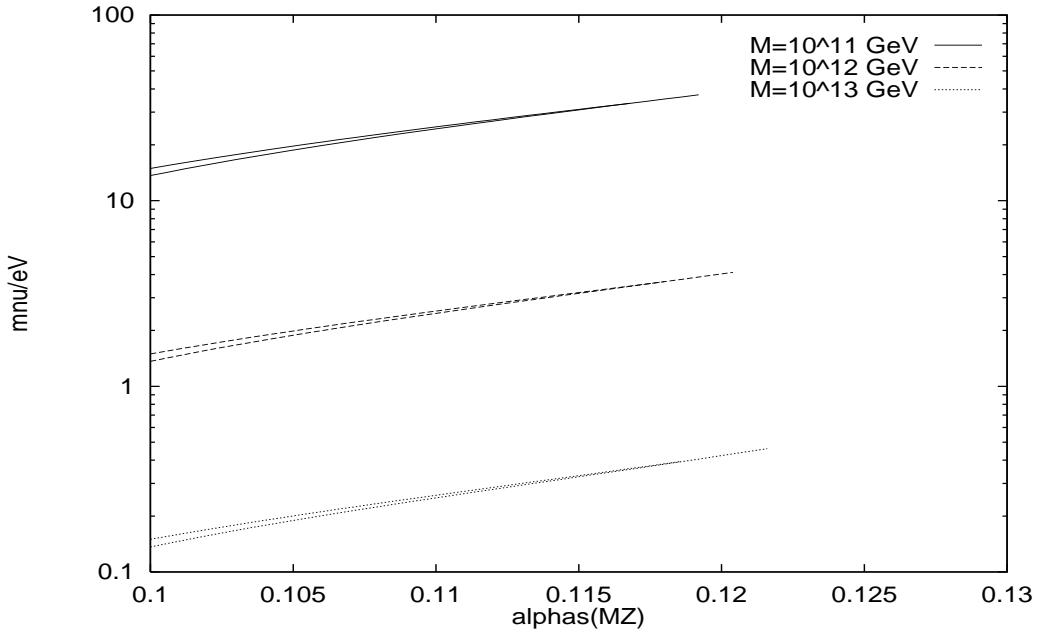


Figure 3: Physical tau neutrino mass predicted for $m_b = 4.25$ GeV and various $M, \alpha_S(M_Z)$. The upper lines correspond to $M_X = 10^{16}$ GeV and the lower lines to $M_X = 10^{17}$ GeV.

or 10^{17} GeV. Because $\tan \beta$ and m_t do not change significantly once the right handed neutrinos has been taken into account, it is reasonable to state that previous third family calculations based on triple Yukawa unification in the MSSM are valid in this scheme also.

Fig.3 shows the physical masses of the tau neutrino for $M = 10^{11,12,13}$ GeV. For $M = 10^{10}$ GeV, $m_{\nu\tau} > 150$ eV and so is excluded on cosmological grounds. In fact, the cosmological bound

$$\sum_{i=e,\mu,\tau} m_{\nu_i} < 100 \text{ eV} \quad (12)$$

translates into a bound on the Majorana mass: $M > 10^{10}$ GeV. Note that $M > 10^{13}$ GeV implies that the tau neutrino would not be massive enough to observe with present experiments. The choice of M_X does not make an appreciable difference to the tau neutrino mass. The effect of the empirical range of $m_b = 4.1 - 4.4$ GeV is shown in Fig.4 for $M = 10^{11}$ GeV. $m_{\nu\tau}$ can vary by up 50 percent at low $\alpha_S(M_Z)$ for different values of $m_b(m_b)$. Similar plots are obtained when M_X is set equal to 10^{17} GeV. Different M values reproduce similar plots, with the mass scaled as in Fig.3.

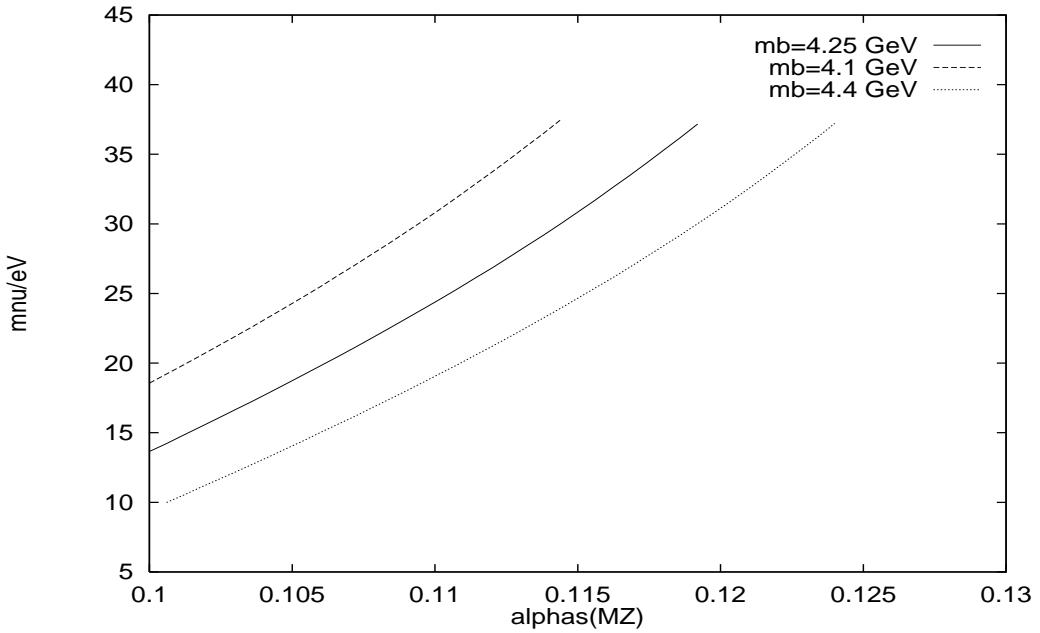


Figure 4: Physical tau neutrino mass predicted for $M = 10^{11}$ GeV, $M_X = 10^{16}$ GeV and various $m_b, \alpha_s(M_Z)$.

The high value of $\tan\beta$ required for triple or quadruple Yukawa unification is not stable under radiative corrections unless some other mechanism such as extra approximate symmetries are invoked. m_t may have been overestimated, since for high $\tan\beta$, the equations for the running of the Yukawa couplings in the MSSM can get corrections of a significant size from Higgsino–stop and gluino–sbottom loops. The size of this effect depends upon the mass spectrum and may be as much as 30 GeV [15]. Not included in our analysis are threshold effects, at low or high energies. These could alter our results by several per cent and so it should be borne in mind that all of the mass predictions have this uncertainty in them. It is also unclear how reliable 3 loop perturbative QCD at 1 GeV is.

In conclusion, we have derived 3 family RGEs for the MSSM that include the neutrino Yukawa coupling. We impose quadruple Yukawa unification and make predictions for m_t , $\tan\beta$ and $m_{\nu\tau}$. The values of m_t and $\tan\beta$ predicted in this scheme are approximately equivalent to results from triple Yukawa unification for values of M that do not violate the cosmological bound on the neutrino masses ($M > 10^{10}$ GeV). A range of $M = 10^{11} - 10^{12}$ GeV predicts a tau neutrino mass that does not violate the cosmological bound in Eq.12 and that could possibly be observed in present day experiments. Is such a value of M reasonable theoretically? A survey has recently

been made of the application of the Witten mechanism [8] to various models [17], where it was seen that usually rather low values of M are found. Interestingly, out of several models examined, supersymmetric $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ gave the highest M generated by the Witten mechanism [17]. In that model, $M \sim M_X/(10^5 - 10^6)$ and so $M = 10^{11}$ GeV could result if $M_X \sim 10^{17}$ GeV. This model [6, 7] also predicts quadruple Yukawa unification and so provides a predictive and simple scheme of viable tau neutrino mass generation.

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